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EFFECT OF THE AXIAL COMPONENT OF THE HEAT FLUX ON SOLIDIFCATION OF A METAL WITH CONTINUOUS CASTING

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In many pieces of work devoted to the theory of the solidification of a metal with continuous casting, as a rule, the axial component of the heat flux is neglected and an approximate equation of thermal conductivity with constant thermophysical parameters of the metal is considered [1-5]. The present article, on the basis of an exact equation of the thermal conductivity, considers the process of the solidification of a continuous ingot, with an arbitrary dependence of the thermophysical parameters of the metal on the temperature. From an analysis of the self-similar solution found, a condition is obtained, with which an approximate consideration of the problem without taking account of the axial component of the heat flux is valid. As an example, let us examine the process of the solidification of a flat aluminum ingot.

We shall postulate that a flat ingot with a thickness of $2x_0$ moves along the Z axis with a constant velocity v. Here we assume that the temperature of the melt (the liquid phase) is equal to the crystallization temperature T_{cr}.

The equation determining the distribution of the temperature T in the solid phase under fully established conditions has the form

$$vc_V\frac{\partial T}{\partial z}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right), \quad (1)$$

where the volumetric heat capacity c_V and the thermal conductivity λ depend on the temperature T.

We write the boundary condition at the cooled surface of the ingot in the form

$$\lambda\frac{\partial T}{\partial x}\Big|_{x=x_0}=q(z), \quad (2)$$

where q(z) is the law of heat removal, whose form will be determined below. Specifically, if the heat removal takes place according to the Newton—Rajchman law. Then, we set

$$q(z)=k(z)\left[T|_{x=x_0}-T_c(z)\right]. \quad (3)$$

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where $k(z)$ is the coefficient of heat transfer from the ingot to the cooling medium, whose value is determined from the solution of the problem with a given law of change in the temperature of the cooling medium $T_c(z)$.

At the crystallization surface $\xi(z)$, the following conditions must be satisfied:

$$\left[\lambda \frac{\partial T}{\partial x} - \lambda \frac{\partial T}{\partial z} \xi'(z)\right]_{x=\xi(z)} = \varkappa \rho v \xi'(z); \tag{4}$$

$$T|_{x=\xi(z)} = T_{cr}; \tag{5}$$

$$\xi(0) = x_0, \quad \xi(h) = 0. \tag{6}$$

The prime in (4) means differentiation with respect to z. We shall seek the solution of the problem in the region $0 \le x \le x_0$, $0 \le z \le h$, where h is the depth of the liquid lune.

We bring Eqs. (1)-(6) into dimensionless form, taking as characteristic values the crystallization temperature T_{cr}, the half-thickness of the ingot x_0, the thermal conductivity λ_{cr}, and the heat capacity c_{Vcr} with $T = T_{cr}$. After going over to the new independent variables

$$y_1 = 1 - x/x_0, \quad u_1 = z/h \tag{7}$$

the dimensionless system of equations assumes the form

$$h_1 \operatorname{Pe} c_{V1} \frac{\partial T_1}{\partial u_1} = h_1^2 \frac{\partial}{\partial y_1}\left(\lambda_1 \frac{\partial T_1}{\partial y_1}\right) + \frac{\partial}{\partial u_1}\left(\lambda_1 \frac{\partial T_1}{\partial u_1}\right) \quad (0 \leqslant y_1, \quad u_1 \leqslant 1); \tag{8}$$

$$\lambda_1 \frac{\partial T_1}{\partial y_1}\Big|_{y_1=0} = \operatorname{Bi}(u_1)(T_1|_{y_1=0} - T_c) = q_1(u_1); \tag{9}$$

$$\left[h_1^2 \frac{\partial T_1}{\partial y_1} + \frac{\partial T_1}{\partial u_1} \xi_1'(u_1)\right]_{y_1=1-\xi_1(u_1)} = -\varkappa_1 h_1 \operatorname{Pe} \xi_1'(u_1); \tag{10}$$

$$T_1|_{y_1=1-\xi_1(u_1)} = 1; \tag{11}$$

$$\xi_1(0) = 1, \; \xi_1(h_1) = 0, \tag{12}$$

where

$$T_1 = T/T_{cr}; \; c_{V1} = c_V/c_{Vcr}; \; \lambda_1 = \lambda/\lambda_{cr};$$
$$q_1 = q(\lambda_{cr} T_{cr}/x_0)^{-1}; \; \xi_1 = \xi/x_0; \; h_1 = h/x_0;$$
$$\varkappa_1 = \varkappa\rho/c_{Vcr} T_{cr}; \; \operatorname{Bi} = kx_0/\lambda_{cr}; \; \operatorname{Pe} = c_{cr} v x_0/\lambda_{cr}.$$

In what follows, the subscript 1 with the dimensionless quantities will be omitted.

We seek the solution of the problem (8)-(12) in the form

$$T = T(\eta), \; \eta = y - u. \tag{13}$$

From (8), taking account of (13), we obtain an equation for determining the function $T(\eta)$,

$$c_V \frac{dT}{d\eta} = -\frac{1 + h^2}{h \operatorname{Pe}} \frac{d}{d\eta}\left(\lambda \frac{dT}{d\eta}\right). \tag{14}$$

The general solution of Eq. (14) has the form

$$\int_1^T \frac{\lambda(t)\, dt}{\int_1^t c_V(\tau)\, d\tau + C_1} = -\frac{h \operatorname{Pe}}{1 + h^2} \eta + C_2. \tag{15}$$

Satisfying conditions (11), (12), we find the constant of integration

$$C_2 = 0 \tag{16}$$

and the form of the liquid lune

$$\xi(u) = 1 - u. \tag{17}$$

We determine the constant C_1 from the condition (9), assigning the values of the temperature of the cooling medium T_c and the heat-transfer coefficient k at the point $u = 0$, which is equivalent to assigning the heat flux along the X axis at the point $y = u = 0$.

We introduce the notation

$$\lambda\partial T/\partial y|_{y=u=0} = \mathrm{Bi}(0)[1 - T_c(0)] = q_0. \tag{18}$$

From (15), taking account of (16), (18), we find

$$C_1 = \frac{1+h^2}{h\,\mathrm{Pe}} q_0. \tag{19}$$

We note that the value of the heat flux $q = \lambda\partial T/\partial y|_{y=1-\xi(u)}$ in the solution in question remains constant along the whole crystallization surface. This follows directly from condition (10) and the solution (17), since $\xi'(u) \equiv -1$.

Substituting the values of the constant C_1 from (19) and C_2 from (16) into formula (15), we find the distribution of the temperature in the skin of the ingot,

$$\int\limits_1^T \frac{\lambda(t)\,dt}{\int\limits_1^t c_V(\tau)\,d\tau - \frac{1+h^2}{h\,\mathrm{Pe}} q_0} = -\frac{h\,\mathrm{Pe}}{1+h^2}\eta. \tag{20}$$

From condition (10) and taking (13), (17), and (20) into account, we obtain an equation determining the depth of the liquid lune,

$$h^2 - \frac{\varkappa\,\mathrm{Pe}}{q_0} h + 1 = 0. \tag{21}$$

From this, taking into consideration that $h \to \infty$ as $\mathrm{Pe} \to \infty$, we find an expression for the depth of the liquid lune,

$$h = \frac{\varkappa\,\mathrm{Pe}}{2q_0} + \sqrt{\left(\frac{\varkappa\,\mathrm{Pe}}{2q_0}\right)^2 - 1}. \tag{22}$$

The steady-state solution obtained exists with

$$\varkappa\mathrm{Pe}/2q_0 \geqslant 1. \tag{23}$$

From (8), (14), and (20), it follows that the axial heat flux due to thermal conductivity can be neglected in Eqs. (1), (4), and (8), (10) with $h^2 \gg 1$. Taking account of the expression for h from (22), this condition assumes the form

$$(\varkappa\mathrm{Pe}/2q_0)^2 \gg 1. \tag{24}$$

Under these circumstances, condition (23) is automatically satisfied, and the depth of the liquid lune, is

$$h = \varkappa\mathrm{Pe}/q_0. \tag{25}$$

For the characteristic of the process of structure formation of a continuous ingot, a knowledge of the values of the crystallization rate and the cooling rate at the crystallization front is very important. The crystallization rate v_{cr} is determined by the expression

$$v_{cr} = v \sin\varphi,$$

where φ is the angle between a tangent to the surface ξ and the Z axis. Since $\sin\varphi = -\xi(z)/\sqrt{1+[\xi'(z)]^2}$ and $\xi'(z) = 1/h$, $\xi'(u) = -1/h$,

$$v_{cr} = v/\sqrt{1+h^2}, \tag{26}$$

where h is determined by relationship (22) and, with satisfaction of condition (24), by relationship (25).

Substituting the expression for h from (22) into (26), and returning to dimensionless values, we write

$$v_{cr} = \sqrt{2}k_0\Delta T/\varkappa\rho_{cr}[1 + \sqrt{1 - (2k_0\Delta T/\varkappa\rho_{cr}v)^2}]^{-1/2}, \tag{27}$$

where $k_0 = k(0)$; $\Delta T = T_{cr} - T_c(0)$.

It follows from (27) that, with an increase in the rate of casting, the crystallization rate rises and, with $v \to \infty$, $v_{cr} \to k_0\Delta T/\varkappa\rho_{cr}$.

The dimensionless cooling rate v_T at the crystallization front is determined by the expression [5]

$$v_T = |\partial T/\partial z|_{x=\xi(z)}. \tag{28}$$

In the solution under consideration $\partial/\partial z = -(1/h)d/d\eta$; consequently, from (28), taking account of (20), we have

$$v_T = q_0/h. \tag{29}$$

From (26), (29), we obtain a relationship connecting the crystallization rate with the cooling rate,

$$v_{cr} = vv_T/q_0\sqrt{1/h^2+1}.$$

For concrete calculations, we assume that the dimensionless values of the thermal conductivity and the heat capacity are linear functions of the temperature

$$\lambda(T) = \mu + \omega T,\ c_V = \nu + \varepsilon T. \tag{30}$$

Substituting (30) into (20) and integrating, we obtain

$$\frac{\varepsilon T + \nu - m}{(\varepsilon T + \nu + m)^n} = \frac{\varepsilon + \nu - m}{(\varepsilon + \nu + m)^n}\exp\left[-\frac{m\varepsilon}{\mu\varepsilon - \omega(\nu - m)}\frac{q_0}{\varkappa}\eta\right], \tag{31}$$

where

$$m = \sqrt{(\nu+\varepsilon)^2 + 2\varkappa\varepsilon};\quad n = \frac{\mu\varepsilon - \omega(\nu+m)}{\mu\varepsilon - \omega(\nu-m)}.$$

Expression (31) describes the law of temperature change in the skin of the ingot if the heat removal takes place according to the law

$$q(z) = q_0\left\{1 - \frac{T_f - 1}{\varkappa}\left[\nu - \frac{\varepsilon}{2}(T_f + 1)\right]\right\},$$

where $T_f = T_f(z)$ is the temperature of the surface of the ingot, determined by formula (31) with $\eta = -z/h$.

For some metals (for example, aluminum, copper), the thermal conductivity and the volumetric heat capacity depend only weakly on the temperature, varying within the limits of ±15% in a rather broad interval of change in the temperature; therefore, with a sufficient degree of accuracy, these parameters can be assumed constant, and their mean values can be taken for calculation. Substituting $\lambda = c_V = 1$ into (20) and integrating, taking account of (22), we find

$$T(\eta) = 1 + \varkappa\left[1 - \exp\left(-\frac{q_0}{\varkappa}\eta\right)\right]. \tag{32}$$

The form of the liquid lune and its depth, as in the nonlinear case, will be determined by formulas (17) and (22), respectively. Here it must be kept in mind that in the linear approximation mean values are taken as characteristic values for the thermal conductivity and the heat capacity, from which it follows that $\lambda = c_V = 1$. The solution (32) is valid if the heat is removed according to the law

$$q(z) = q_0\exp\left(\frac{q_0}{\varkappa h}z\right). \tag{33}$$

From condition (9), taking account of (7) and (33), we find the law of change of Bi(z) [or the heat-transfer coefficient k(z)] with a given temperature $T_c(z)$ of the cooling medium:

$$\mathrm{Bi}(z) = \frac{q_0}{T_f(z) - T_c(z)}\exp\left(\frac{q_0}{\varkappa h}z\right),$$

where T_f is determined from (32) and (7) with x = 1. Since, in the problem under consideration, the temperature of the surface of the ingot cannot be lower than the temperature of the cooling medium, the following condition must be satisfied:

$$T_c(z) \leqslant T_f(z) = 1 + \varkappa\left[1 - \exp\left(\frac{q_0}{\varkappa h}z\right)\right].$$

From this, setting z = h, we find the limitation on the value of the heat flux, with which the solution obtained exists. We have

$$q_0 \leqslant \varkappa\ln\left[1 + \frac{1 - T_c(h)}{\varkappa}\right].$$

As an example, we give the results of a calculation of the solidification of an aluminum ingot with a thickness of 0.2 m, with c_V = 2715 kJ/(m³·°K), λ = 222 W/(m·°K), ρ = 2.6 · 10³

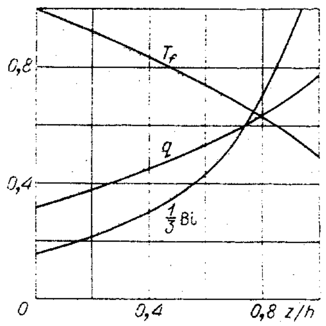

Fig. 1

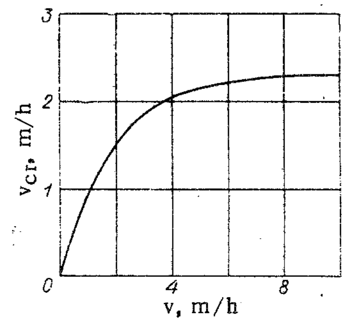

Fig. 2

kg/m^3, $\varkappa$ = 402 kJ/kg, T_{cr} = 933°K, v = 2.78 · 10^{-3} m/sec (Pe = 3.4), k(0) = 936 W/(m^2·°K) [Bi(0) = 0.422]. For the dimensionless temperature of the cooling medium we take the following law of change:

$$T_c(z) = 0.336 - 0.024z/h.$$

Figure 1 gives the dependence of the dimensionless values of the ingot surface temperature T_f, the heat flux q (removed from the ingot), and the Bi number on the variable z.

Figure 2 illustrates the dependence of the crystallization rate on the rate of pulling of the ingot. Here the depth of the liquid lune, according to formula (22), is equal to 0.481 m, while, without taking acocunt of the axial heat flux [Eq. (25)], h = 0.503 m; i.e., the difference is 4.5%.

We note that the solution obtained, in addition to its independent importance, can be used as a test for the verification of numerical algorithms for calculation of the problem of the solidification of a continuous casting of flat geometry, taking account of the dependence of the thermophysical parameters of the metal on the temperature.

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